A MIXED HEAT CONDUCTION BOUNDARY PROBLEM FOR A SEMI-INFINITE PLATE

A. D. Khanzhov

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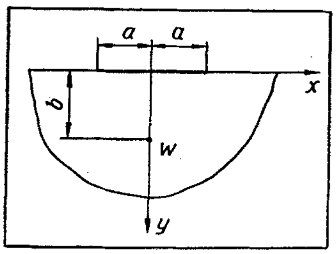
An examination is made of a plane stationary heat conduction problem with mixed conditions assigned on the edge of a semi-infinite plate. There is a temperature field in the plate due to the action of a point source of heat located inside the plate.

The problem is solved in a rectangular system of coordinates, the y axis being perpendicular to the plate boundary. A source of heat of intensity W is located at the point with coordinates x = 0, y = b (see the figure).

In this paper the temperature field is determined for the case when the heat flux is given at the boundary of the plate in a certain region, and the temperature is given outside this region. The base of the plate is thermally insulated.



Location of a heat source W in the coordinate system referred.

The integral transformation method is used to solve the problem.

We require to solve the differential equation of heat conduction

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = -\frac{W}{k}\delta(x)\delta(y-b) \qquad (1)$$

with boundary condition

$$T(x,\ 0) = T_0(x) \ \text{for} \ |x| > a,$$

$$\frac{\partial T(x,\ 0)}{\partial y} = G(x) \ \text{for} \ -a \leqslant x \leqslant a, \qquad (2)$$

where $T_0(x)$ and $G(x)$ are given functions.

Applying a Fourier transformation [1] to Eq. (1) with respect to x and taking account of the condition at infinity, we obtain an inhomogeneous differential equation of second order for the function $\overline{T}$; this having the solution

$$\overline{T}_1(\xi,\ y) = \frac{Q}{2|\xi|}\exp[-(b-y)|\xi|] + B\exp(b-y)|\xi| \ \text{for} \ 0 \leqslant y \leqslant b,$$

$$\overline{T}_2(\xi,\ y) = (Q/2|\xi| + B)\exp[-(y-b)|\xi|] \ \text{for} \ y \geqslant b. \qquad (3)$$

Here D is a function determined from the boundary conditions; $\overline{T}(\xi, y)$ is the Fourier transform of the function T(x, y); $Q = W/2\pi k^{1/2}$.

Applying the inverse theorem for the Fourier transformation for the first equation (3), and differentiating the result with respect to y, we satisfy the boundary conditions (2).

Thus, we arrive at the dual integral equations

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{d\overline{T}_1(\xi,\ 0)}{dy}\exp(-i\xi x)\,d\xi = G(x) \ \text{for} \ -a \leqslant x \leqslant a,$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{T}_1(\xi,\ 0)\exp(-i\xi x)\,d\xi = T_0(x) \ \text{for} \ |x| > a. \qquad (4)$$

Because of symmetry of boundary conditions and of the location of the concentrated heat source with respect to the y axis, the functions $T_1(x, 0)$ and $[dT_1(x, 0)]/dy$ will be even functions with respect to x in the interval $(-\infty, \infty)$. Therefore, the Fourier cosine transform of the function $T_1(x, 0)$ and $[dT_1(x, 0)]/dy$ will coincide with the Fourier transform of these functions.

Taking account of the relation

$$\cos\xi x = (\pi\xi x/2)^{1/2}J_{-1/2}(\xi x)$$

and introducing the new variables

$$\rho = x/a, \quad u = a\xi,$$

the dual integral Eqs. (4), after some calculations, take the form

$$\int_0^\infty ug(u)J_{-1/2}(\rho u)\,du = h(a\rho) \ \text{for} \ 0 \leqslant \rho \leqslant 1,$$

$$\int_0^\infty g(u)J_{-1/2}(\rho u)\,du = 0 \ \text{for} \ \rho > 1. \qquad (5)$$

Here

$$g(u) = \frac{1}{a^2}\left[\left(\frac{u}{a}\right)^{\frac{1}{2}}B\exp\frac{bu}{a} - \frac{u}{a}\overline{\beta}\left(\frac{u}{a}\right)\right]; \qquad (6)$$

$\overline{\beta}(u/a)$ is a Hankel transform of order 1/2 of the function

$$\beta(a\rho) = \frac{\sqrt{a^3}T_0(a\rho)}{\sqrt{\rho}} - \frac{\sqrt{a^3}Q}{2}\int_0^\infty\frac{\exp(-bu/a)}{u^{\frac{1}{2}}}J_{-1/2}(\rho u)\,du; \qquad (7)$$

$$h(a\rho) = -\frac{G(a\rho)}{\sqrt{a\rho}} + \left(\frac{2}{\pi a\rho}\right)^{\frac{1}{2}}\frac{Qb}{b^2+a^2\rho^2} -$$

$$-\frac{1}{a^3}\int_0^\infty u^2\bar{\beta}_1\left(\frac{u}{a}\right)J_{-1/2}(\rho u)\,du; \qquad (8)$$

$\bar{\beta}_1(u)/a$ is a Hankel transform of order 1/2 of the function $a^{3/2}\,T_0(a\rho)/\rho^{1/2}$.

Using the solution of reference [2], we find the desired function

$$g(u) =$$

$$=\left(\frac{2u}{\pi}\right)^{\frac{1}{2}}\int_0^1 \mu^{\frac{3}{2}}J_0(\mu u)\,d\mu\int_0^1 h(\mu\psi)\,\psi^{\frac{1}{2}}(1-\psi^2)^{-\frac{1}{2}}\,d\psi. \qquad (9)$$

Returning to the old variables and using formulas (3), (6), and (9), and the inversion theorem for the Fourier cosine transformation, we find the stationary temperature field in the semi-infinite plate:

$$T_1(x,\ y)=\sqrt{2/\pi}\int_0^\infty \bar{T}_1(\xi,\ y)\cos\xi x\,d\xi \quad \text{for} \quad 0\leqslant y\leqslant b,$$

$$T_2(x,\ y)=\sqrt{2/\pi}\int_0^\infty \bar{T}_2(\xi,\ y)\cos\xi x\,d\xi \quad \text{for} \quad y\geqslant b. \qquad (10)$$

We shall examine the case when a section $-a \le x \le a$ is thermally insulated, or the temperature is equal to zero at another section $|x| > a$ on the boundary of the plate at $y = 0$:

$$\frac{\partial T(x,\ 0)}{\partial y}=0 \quad \text{for} \quad -a\leqslant x\leqslant a,$$

$$T(x,\ 0)=0 \quad \text{for} \quad |x|>a.$$

In this case

$$h(a\rho)=\left(\frac{2}{\pi a\rho}\right)^{\frac{1}{2}}\frac{Qb}{b^2+a^2\rho^2};$$

$$g(u)=\left(\frac{u}{a}\right)^{\frac{1}{2}}Q\int_0^1\frac{\mu J_0(\mu u)}{\sqrt{b^2+a^2\mu^2}}\,d\mu.$$

Formulas (10) in this case give

$$T_1(x,\ y)=T_2(x,\ y)=T(x,\ y);$$

$$T(x,\ y)=\frac{W}{2\pi k}\left[\frac{1}{2}\ln\frac{(b+y)^2+x^2}{(b-y)^2+x^2}+\right.$$

$$\left.+\ln\left|\frac{R+a^2+b^2+\sqrt{2(a^2+b^2)(y^2-x^2+a^2+R)}}{(b+y)^2+x^2}\right|\right],$$

where

$$R=\sqrt{(y^2-x^2+a^2)^2+4x^2y^2}.$$

The formula is valid both for $0 \le y \le b$, and for $y \ge b$. Putting $a = 0$, we obtain the well known solution [3]

$$T(x,\ y)=\frac{W}{4\pi k}\ln\frac{(b+y)^2+x^2}{(b-y)^2+x^2}.$$

NOTATION

k is the thermal conductivity of the plate; T is the plate temperature; W is intensity of the heat source; δ is the Dirac delta function; ξ is the integral transformation parameter; $J_{-1/2}$ is the Bessel function of the first kind of order $-1/2$; J_0 is the Bessel function of the first kind of zero order.

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Polytechnic Institute,
Saratov